Assignment 9.

Taylor Series.

This assignment is due Wednesday, March 30. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. This assignment is long[1], pace yourself.

In this homework, you can take for granted basic inequalities involving limsup and arithmetic operations. You also can take for granted that $\sqrt[n]{\frac{n!}{n^n}} \to \frac{1}{e}$ as $n \to \infty$.

(1) In this problem we look into single-valued versions of complex logarithm and z^α. Let $G = \mathbb{C} \setminus (-\infty, 0]$, that is the complex plane without the non-positive part of real axis.

(a) For $z \in G$, define

$$\ln z = \ln|z| + i \arg z.$$

(In particular, $-\pi < \operatorname{Im} \ln z < \pi$.) Show that $\ln z$ is analytic on the domain G, and find its derivative (among other things, you can use Problem 5 of HW3).

(b) Use Taylor series theorem to find Taylor series for $\ln(1 + z)$ at $z = 0$. What is the radius of convergence?

(c) For $\alpha \in \mathbb{C}$ and $z \in G$, define

$$z^\alpha = \exp(\alpha \ln z).$$

Argue that z^α is analytic on G and find its derivative. (*Hint:* Use chain rule.)

(d) Use Taylor series theorem to find Taylor series for $(1 + z)^\alpha$ at $z = 0$. What is the radius of convergence?

Remark. It is convenient to denote $\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ by $\binom{\alpha}{n}$.

(2) Find the radius of convergence of each of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty} n^k z^n$ $(k = 0, 1, 2, \ldots)$; (b) $\displaystyle\sum_{n=1}^{\infty} n^n z^n$; (c) $\displaystyle\sum_{n=1}^{\infty} (2+(-1)^n)^n z^{2n}$;

(d) $\displaystyle\sum_{n=1}^{\infty} (\cos in) z^n$; (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^k}{n!} z^n$; (f) $\displaystyle\sum_{n=1}^{\infty} (n + a^n) z^n$ $(a \in \mathbb{C})$;

(g) $\displaystyle\sum_{n=1}^{\infty} 3^n z^{n^2}$; (h) $\displaystyle\sum_{n=1}^{\infty} 2^n z^{n!}$.

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(3) Given that the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n z^n$ is R $(0 < R < \infty)$, what is the radius of convergence of the following series?

(a) $\sum_{n=1}^{\infty} n^k c_n z^n$ $(k = 0, 1, 2, \ldots)$; (b) $\sum_{n=1}^{\infty} (2^n - 1) c_n z^n$; (c) $\sum_{n=1}^{\infty} \frac{c_n}{n!} z^n$;

(d) $\sum_{n=1}^{\infty} c_n^k z^n$ $(k = 0, 1, 2, \ldots)$; (e) $\sum_{n=1}^{\infty} c_n z^{2n}$; (f) $\sum_{n=1}^{\infty} c_n z^{n^2}$.

(4) Let the radii of convergence of the power series $\sum_{n=1}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} b_n z^n$ equal R_1 and R_2, respectively. Let R be the radius of convergence of

(a) $\sum_{n=1}^{\infty} (a_n + b_n) z^n$. Prove that $R \geq \min(R_1, R_2)$.

(b) $\sum_{n=1}^{\infty} a_n b_n z^n$. Prove that $R \geq R_1 R_2$.

(*Hint:* In (a), it's easier to argue by basic properties of convergence. In (b), by Cauchy–Hadamard.)

(5) In this problem, we figure out whether the inequality in Problem 4a can be made an equality.

(a) Give an example of two power series $\sum_{n=1}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} b_n z^n$ with the same finite radius of convergence $R_1 = R_2$, such that the radius of convergence of $\sum_{n=1}^{\infty} (a_n + b_n) z^n$ is infinite.

(b) If $R_1 < R_2$, can the radius of convergence of $\sum_{n=1}^{\infty} (a_n + b_n) z^n$ be strictly greater than R_1? (*Hint:* Argue by basic properties of convergence.)

(6) Give an alternative prove of Liouville's theorem, based on the use of Cauchy integral formula as follows. Considering the integral

$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz,$$

where $a \neq b$, $|a| < R$, $|b| < R$, and take $R \to \infty$.